Algorithms for Handwritten Digit Recognition

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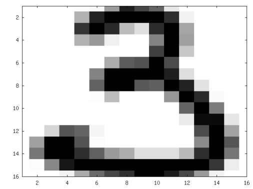
The Problem

Automatically classify a single unknown handwritten digit using a database of known digits.

An Unknown Digit (Test Image)

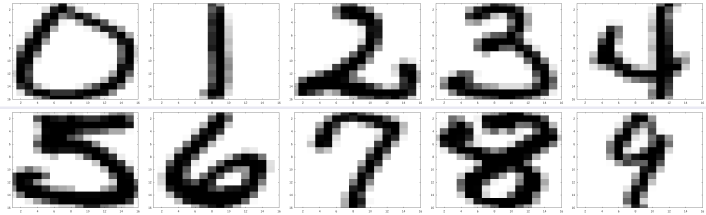


0? … 2? … 3? … 5? … 9?

- 16×16-pixel grayscale images (matrices) of digits $0, \ldots, 9$.
- Application: Automatic mail sorting at the post office.

Image Representation

Images from the Database



- Scanned and rescaled to 16×16-pixel grayscale images.
- Pixels take floating point values between -1 (white) and 1 (black).

The Database

- 7291 handwritten digits collected by the U.S. Postal Service. [1]

Breakdown of Digits

Digit	Sample Size
0	1194
1	1005
2	731
3	658
4	652
5	556
6	664
7	645
8	542
9	644

[1] Database retrieved from http://www-stat.stanford.edu/~tibs/ElemStatLearn/data.html

The Algorithms

We try two classification algorithms.

- Singular Value Decomposition Based Algorithm
- Tangent Distance Algorithm

SVD Based Algorithm

First Algorithm:
The SVD Based Algorithm

How We Handle the Database

- Unroll the 16×16-pixel images into vectors in $\mathbb{R}^{256}$.
- Collect all the different types (0 through 9) of unrolled images.
- Place all unrolled images of type $i \in \{0, 1, ..., 9\}$ into the matrix D_i as the columns.

$$D_5 = \begin{bmatrix} | & | & | & \dots & | \\ 5 & 5 & \mathit{5} & \dots & \mathbf{5} \\ | & | & | & \dots & | \end{bmatrix}$$

$$D_5 \in \mathbb{R}^{256 \times 556}$$

Notice there are many more columns than rows.

The Column Space and Least Squares

Take a test image $d \in \mathbb{R}^{256}$.

$$D_5 = \begin{bmatrix} | & | & | & \dots & | \\ 5 & 5 & \mathit{5} & \dots & \mathbf{5} \\ | & | & | & \dots & | \end{bmatrix}, \qquad d = ?$$

- Is d a linear combination of the columns of some D_i?
- How close is d to being a linear combination of the columns of D_i?

Solve a least squares problem!

$$\rho_i = \min_x \|D_i x - d\|_2^2$$

Observe: We are interested in the residual ρ_i and not the x.

A Classification Algorithm

Let $d \in \mathbb{R}^{256}$ be a test digit to classify and let $i \in \{0, 1, ..., 9\}$.

- Form the D_i matrices (as described before) for every i.
- For every i, find $\rho_i = \min_x \|D_i x - d\|_2^2$.
- Compute $\min_i\{\rho_i\}$ and classify d as a digit of type "i".

The residual can be found by using brute force to solve the least squares problem, but this is expensive. We will show how to implement this algorithm efficiently by using properties of the SVD.

Singular Value Decomposition Theorem

Theorem (Singular Value Decomposition)

Let $A \in \mathbb{R}^{m \times n}$ be a nonzero matrix with rank r. Then A can be expressed as a product $A = U\Sigma V^T$, where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are orthogonal, and $\Sigma \in \mathbb{R}^{m \times n}$ is a "diagonal" matrix with diagonal entries (called singular values) $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0 = \sigma_{r+1} = \ldots = \sigma_n$. Furthermore, the columns of U (called singular vectors) corresponding to nonzero singular values form an orthogonal basis for the column space of A.

$$U = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix}, \Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & \sigma_r & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

SVD and Least Squares

Consider the following least squares problem for $A \in \mathbb{R}^{m \times n}$ of rank r where $n >> m$. The residual ρ is given by

$$\rho = \min_x \|Ax - d\|_2^2 \quad \Leftrightarrow \quad A^T Ax = A^T d.$$

(Notice that Ax is in the column space of A)

Using the SVD $A = U\Sigma V^T$ and the fact that the first r columns of U span the column space of A

$$\rho = \min_y \|U_r y - d\|_2^2 \quad \Leftrightarrow \quad U_r^T U_r y = U_r^T d \quad \Leftrightarrow \quad y = U_r^T d.$$

Substituting gives an easy formula for computing the residual

$$\rho = \|U_r U_r^T d - d\|_2^2.$$

Reducing Computation Requirements

$$\rho = \| U_r U_r^T d - d \|_2^2$$

$$1 \leq r \leq 256$$

The formula for the residual is nice, but for large values of r computation time and storage requirements are high.

One way to make it more efficient is by approximation.

SVD Approximation Theorem

The following theorem allows us to make the best possible rank k approximation of a matrix A. For our purposes $k << r$ (low rank approximation).

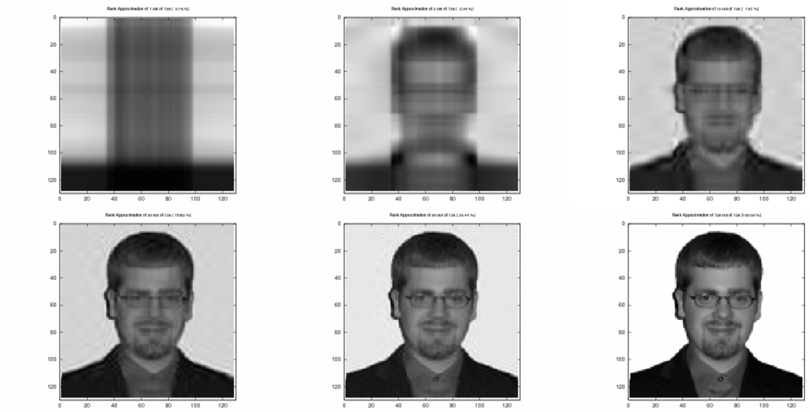
Theorem (SVD Approximation)

Let $A \in \mathbb{R}^{m \times n}$ be a nonzero matrix with rank r. Let $\sigma_1, \ldots, \sigma_r$ be the singular values of A, with associated left and right singular vectors $u_1, \ldots, u_r$ and $v_1, \ldots, v_r$, respectively, and let $k \leq r$. Then $A = U\Sigma V^T = \sum_{j=1}^{r} \sigma_j u_j v_j^T$, and $A_k = \sum_{j=1}^{k} \sigma_j u_j v_j^T$ is the best rank k approximation for A under the 2-norm.

What other uses does the theorem have?

SVD Image Compression

- 128×128 image.
- Left to right: (top) rank 1, 3, 10, (bottom) 20, 30, 98 (full).



Proof of the SVD Approximation Theorem

Proof.

It's clear that $A_k = \sum_{j=1}^{k} \sigma_j u_j v_j^T$ has rank k. Computing $\|A - A_k\|_2$ we have

$$\|A - A_k\|_2 = \left\| \sum_{i=k+1}^{n} \sigma_i u_i v_i^T \right\|_2 = \|U\Sigma_{k+1}V^T\|_2 = \|\Sigma_{k+1}\|_2 = \sigma_{k+1}.$$

Let B be a rank k $m \times n$ matrix, so it's null space has dimension $n - k$. The space spanned by $\{v_1, ..., v_{k+1}\}$ has dimension $k + 1$. Since $(n - k) + (k + 1) > n$, the intersection of $\mathcal{N}(B)$ and $\{v_1, ..., v_{k+1}\}$ must be non-trivial. Let h be a unit vector in their intersection. Then $h = c_1 v_1 + \cdots + c_{k+1} v_{k+1} = V_{k+1} c$ with $\|h\|_2 = 1$.

$$\|A - B\|_2 \geq \|(A - B)h\|_2 = \|Ah\|_2 = \|U\Sigma V^T h\|_2 = \|\Sigma(V^T h)\|_2$$

$$= \|\Sigma(V^T V_{k+1} c)\|_2 = \left\| \Sigma \begin{bmatrix} I_{k+1} \\ 0 \end{bmatrix} c \right\|_2 \geq \sigma_{k+1} \|c\|_2 = \sigma_{k+1}.$$

$\square$

Developing the SVD Based Algorithm

Corollary

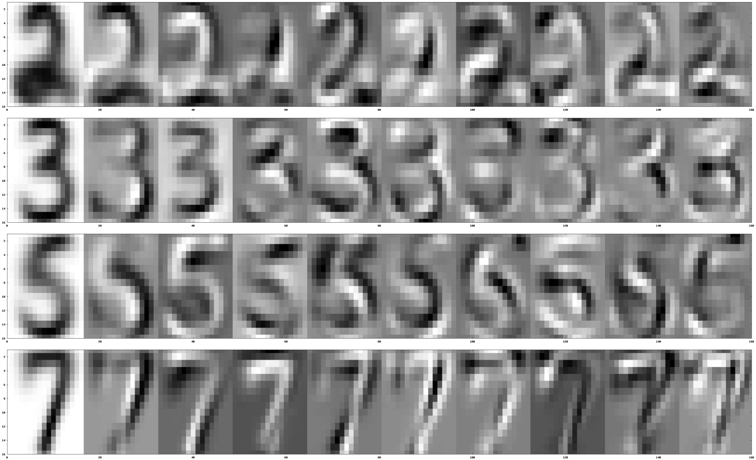
Let $A \in \mathbb{R}^{m \times n}$ be a nonzero matrix of rank r with singular value decomposition $A = U \Sigma V^T$. Then the first $k < r$ columns $u_1, \ldots, u_k$ of U form an orthogonal basis for the column space of A_k. Furthermore, $U_k = [u_1 \quad u_2 \quad \ldots \quad u_k]$ implies $U_k^T U_k = I$.

Proof.

Let $A_k = \sum_{j=1}^{k} \sigma_j u_j v_j^T$ and $y \in \mathbb{R}^n$. Observe that $A_k y = \sum_{j=1}^{k} \sigma_j u_j (v_j^T y)$. This means $u_1, \ldots, u_k$ form an orthogonal basis for the column space of A_k. For the second part, notice that the rows of matrix U_k^T are orthogonal to the columns of U_k which means $U_k^T U_k = I$. □

The Singular Vectors of the Database

$u_1, \ldots, u_{10}$ for D_2, D_3, D_5, D_7.



Developing the SVD Based Algorithm (cont.)

Corollary

Let $A \in \mathbb{R}^{m \times n}$ be a nonzero matrix of rank r with a rank k approximation A_k. The least squares problem $\min_x \|U_k x - d\|_2^2$ has the solution $x = U_k^T d$ with residual $\|U_k U_k^T d - d\|_2^2$.

Before we used A to find the residual.

$$\rho = \min_x \|Ax - d\|_2^2 \quad \Rightarrow \quad \rho = \|U_r U_r^T d - d\|_2^2$$

(Notice that Ax is in the column space of A)

Now we use A_k to approximate the residual.

$$q = \min_x \|A_k x - d\|_2^2 \quad \Rightarrow \quad q = \|U_k U_k^T d - d\|_2^2$$

(Notice that $A_k x$ is in the column space of A_k)

Why Do We Use Fixed Low Rank Approximation?

- Reduces the computation time (pre-compute U_k).
- Gives a cheap formula for the residual.
- Avoids disasters (some D_i matrices span $\mathbb{R}^{256}$!).
- Provides fairness (not all D_i matrices have the same rank).

The SVD Based Algorithm

Let $i \in \{0, 1, ..., 9\}$.

Do once at startup:

- Form the D_i matrices for every i.
- Compute the SVD of each D_i.
- Do a rank k approximation of each D_i and store each U_{i_k}.

Let $d \in \mathbb{R}^{256}$ be a test digit to classify.

- For every i, compute $q_i = \|U_{i_k} U_{i_k}^T d - d\|_2^2$.
- Compute $\min_i\{q_i\}$ and classify d as an "i".

SVD Based Algorithm Results

The following data are the test results for the SVD based algorithm with a rank approximation of 10 on a sample of 2007 test digits.

Digit	Sample Size	Correct	Incorrect	Success Rate
0	359	353	6	98.329%
1	264	260	4	98.485%
2	198	179	19	90.404%
3	166	143	23	86.145%
4	200	183	17	91.500%
5	160	145	15	90.625%
6	170	160	10	94.118%
7	147	138	9	93.878%
8	166	149	17	89.759%
9	177	168	9	94.915%

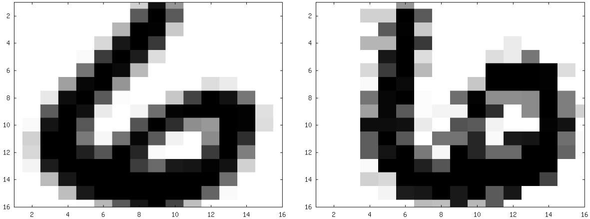
Average Success Rate: 93.572%. Run time: 76 seconds.

Tangent Distance Algorithm

Second Algorithm:
Tangent Distance Algorithm

Transformations and Euclidean Distance

Consider rotating a digit p by an angle α_p.

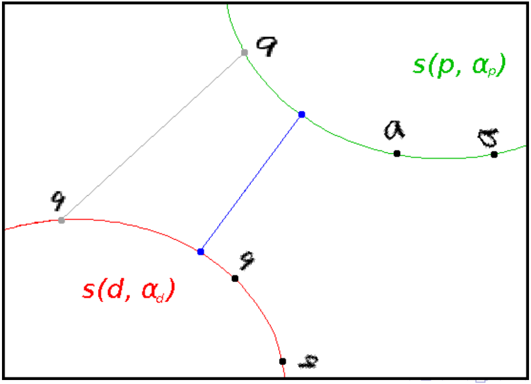


Using the vector form of p (i.e. $p \in \mathbb{R}^{256}$), we can represent all rotations of the digit p by a parameterized curve $s(p, \alpha_p) \subset \mathbb{R}^{256}$ where α_p is the angle of rotation. Notice $s(p, 0) = p$.

Graphs of the Curves

The original distance between the curves and the minimum distance between the parameterized curves (impossible to compute).

$$min_{\alpha_p,\alpha_d} \|s(p,\alpha_p) - s(d,\alpha_d)\|_2^2 \qquad \downarrow \mathbb{R}^{256}$$



Taylor Series Approximation

The the equation for the parameterized curve $s(p, \alpha_p)$ is unknown and nonlinear, but can be approximated by Taylor expansion

$$s(p, \alpha_p) = s(p, 0) + \frac{ds}{d\alpha_p}(p, 0)\alpha_p + \mathcal{O}(\alpha_p^2) \approx p + t_p\alpha_p$$

where $t_p = \frac{ds}{d\alpha}(p, 0) \in \mathbb{R}^{256}$.

Now consider a test digit $d \in \mathbb{R}^{256}$ to classify and a parameterized curve for it.

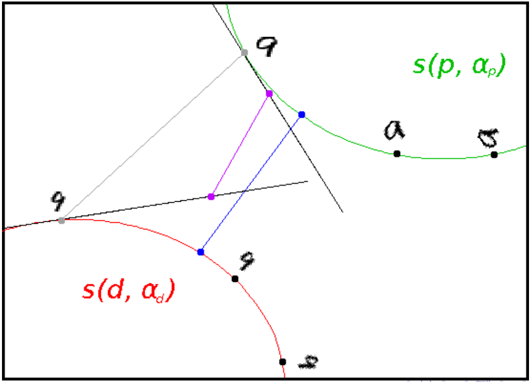
$$s(d, \alpha_d) = s(d, 0) + \frac{ds}{d\alpha_d}(d, 0)\alpha_d + \mathcal{O}(\alpha_d^2) \approx d + t_d\alpha_d$$

Good Thing: We have *linear* approximations for $s(p, \alpha_p)$ and $s(d, \alpha_d)$.

What is Tangent Distance?

The tangent distance is an approximation to the minimum distance between the parameterized curves.

$$\min_{\alpha_p,\alpha_d} \|s(p,\alpha_p) - s(d,\alpha_d)\|_2^2 \approx \min_{\alpha_p,\alpha_d} \|(p + t_p\alpha_p) - (d + t_d\alpha_d)\|_2^2.$$



Computing Tangent Distance

We can approximate the distance between the two curves by the tangent distance.

$$\min_{\alpha_p,\alpha_d} \|s(p,\alpha_p) - s(d,\alpha_d)\|_2^2 \approx \min_{\alpha_p,\alpha_d} \|(p + t_p\alpha_p) - (d + t_d\alpha_d)\|_2^2$$

$$= \min_{\alpha_p,\alpha_d} \|(p-d) - (-t_p \quad t_d)(\alpha_p \quad \alpha_d)^T\|_2^2 = t_{pd}.$$

This is to say that finding the tangent distance t_{pd} between p and d is the same as solving this least squares problem.

Multivariate Parameters

Consider doing k transformations (rotation, scaling, translation, ...) on a digit p. How will the tangent distance change?

Now $s(p, a_p) \subset \mathbb{R}^{256}$ with $a_p = (\alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_k)^T$. We can find a multivariate Taylor expansion for $s(p, a_p)$

$$s(p, a_p) = s(p, 0) + \sum_{i=1}^{k} \frac{\partial s}{\partial \alpha_i}(p, 0)\alpha_i + \mathcal{O}(\|a_p\|_2^2) \approx p + T_p a_p$$

$$T_p = \begin{pmatrix} \frac{\partial s}{\partial \alpha_1} & \frac{\partial s}{\partial \alpha_2} & \dots & \frac{\partial s}{\partial \alpha_k} \end{pmatrix}$$

Computing Tangent Distance in the Multivariate Case

How does computing the tangent distance change in the multivariate case?

$$\min_{a_p, a_d} \|(p + T_p a_p) - (d + T_d a_d)\|_2^2$$

$$= \min_{a_p, a_d} \|(p - d) - (-T_p \quad T_d)(a_p \quad a_d)^T\|_2^2 = t_{pd}.$$

It's still a least squares problem! Now the question is what exactly are T_p and T_d?

$$T_p = \begin{pmatrix} \frac{\partial s}{\partial \alpha_1} & \frac{\partial s}{\partial \alpha_2} & \cdots & \frac{\partial s}{\partial \alpha_k} \end{pmatrix}$$

Answer: T_p is a Jacobian matrix consisting of derivatives of transformations evaluated at $(p, 0)$. T_d is the same thing for d at $(d, 0)$.

Derivatives of Transformations

Let $f(x, y) \in \mathbb{R}$ be a differentiable function such that for a digit matrix $P \in \mathbb{R}^{16\times16}$, $f(i, j) = P_{ij}$ for all $i, j \in \{1, 2, ..., 16\}$ (e.g. $f(3, 4) = P_{3,4}$).

$$P \Leftrightarrow p \qquad \mathbb{R}^{16\times16} \Leftrightarrow \mathbb{R}^{256}$$

The derivatives of the transformations at $\alpha = 0$ are

Translation in the x direction	f_x
Translation in the y direction	f_y
Rotation about the "origin"	$yf_x - xf_y$
Scaling	$xf_x + yf_y$
Stretch/compress along the horizontal and vertical axes	$xf_x - yf_y$
Stretch/compress along the diagonals	$yf_x + xf_y$
Thickening	$(f_x)^2 + (f_y)^2$

Derivation of the Scaling Derivative

Let $f(x, y)$ be as before. Scaling is achieved by

$$s(p, \alpha_s)(x, y) = f((1 + \alpha_s)x, (1 + \alpha_s)y).$$

Using the chain rule to differentiate and evaluating at $\alpha_s = 0$ we get

$$\frac{d}{d\alpha_s}(s(p, \alpha_s)(x, y))|_{\alpha_s=0} = xf_x + yf_y.$$

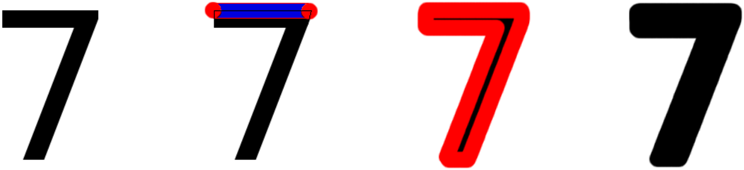
The derivation of the other derivatives is mostly the same with the exception of the thickening derivative.

Thickening Derivative

Let $f(x, y)$ be as before. The thickened image is obtained by defining a new function

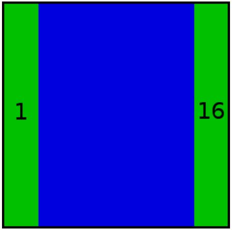
$$g_\alpha(x, y) = \max_{\|r\|<\alpha} f(x + r_1, y + r_2)$$

where $r = (r_1, r_2)$ is a vector in $\mathbb{R}^2$.



For a complete derivation and discussion see Simard et. al.

How to Compute f_x and f_y



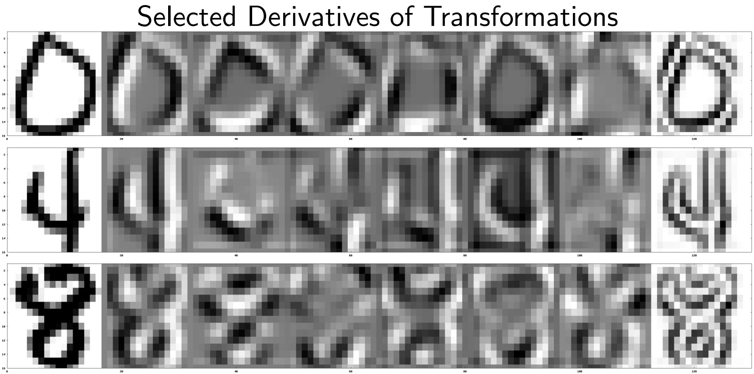
$$f_x(i,j) \approx \frac{f(i+1,j) - f(i-1,j)}{2}$$

$$f_x(1,j) \approx \frac{f(2,j) - f(1,j)}{1}$$

$$f_x(16,j) \approx \frac{f(16,j) - f(15,j)}{1}$$

- All derivatives can be formed from f_x and f_y.
- Compute f_x and f_y using finite differences.
- Use two-sided finite differences in the interior.
- Use one-sided finite differences at the ends.

Derivatives for Selected Digits



Listed Left to Right: Original image, x-translation, y-translation, rotation, scaling, stretch/compress along horizontal/vertical, stretch/compress along diagonals, thickening.

The Tangent Distance Algorithm

Do once at startup:

- Construct the tangent matrix T_p for every p in the database.

Let $d \in \mathbb{R}^{256}$ be a test digit to classify.

- Construct the tangent matrix T_d.
- Compute the tangent distance t_{pd} between d and every p.
- Find $r = \min_p\{t_{pd}\}$.
- Classify d as the digit corresponding to the p that gives r.

Tangent Distance Algorithm Results

The following data are the test results for the tangent distance algorithm on a sample of 2007 test digits.

Digit	Sample Size	Correct	Incorrect	Success Rate
0	359	289	70	80.501%
1	264	255	9	96.591%
2	198	172	26	86.869%
3	166	145	21	87.349%
4	200	145	55	72.500%
5	160	143	17	89.375%
6	170	161	9	94.706%
7	147	137	10	93.197%
8	166	130	36	78.313%
9	177	166	11	93.785%

Average Success Rate: 86.846%. Run time: 25.5 hours. ($7291 \times 2007 = 14{,}633{,}037$).

Closing Remarks

Closing Remarks

Summary of Results

Below are the test results for both algorithms. [2]

SVD Based Algorithm with Rank Approximation of 10:

- Accuracy: 93.5%
- Run time: 76 seconds
- Suited for real time.

Tangent Distance Algorithm:

- Accuracy: 86.8% (91%)
- Run time: 25.5 hours (18 hours)
- Suited for "tie-breaking" (smallest ρ_i is close to another).

Omitting the thickening transformation yielded the numbers in parentheses.

[2] The testing platform was a 2.4 GHz AMD Athlon 64 X2 processor machine with 2 GB of memory running Debian GNU/Linux. The software used to test the algorithms was Octave 3.0 running on a single core.

Further Reading

Further reading about the two algorithms.

- L. Eldén. *Matrix Methods in Data Mining and Pattern Recognition*. SIAM, Philadelphia, 2007; 113-122.
- B. Savas. *Analyses and Tests of Handwritten Digit Algorithms*. Master's thesis, Mathematics Department, Linköping University, 2002.
- P.Y. Simard, Y.A. Le Cun, J.S. Denker and B. Victorri. Transformation invariance in pattern recognition - tangent distance and tangent propagation. *International Journal of Imaging Systems and Technology*, 2001; 181-194.

The End!